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Virasoro symmetry of constrained KP hierarchies

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Abstract

The conventional formulation of additional nonisospectral symmetries for the full Kadomtsev–Petviashvili (KP) integrable hierarchy is not compatible with the reduction to the important class of constrained KP (cKP) integrable models. This paper solves explicitly the problem of compatibility of the Virasoro part of additional symmetries with the underlying constraints of cKP hierarchies. Our construction involves an appropriate modification of the standard additional-symmetry flows by adding a set of "ghost symmetry" flows. We also discuss the special case of cKP – truncated KP hierarchies, obtained as Darboux–Bäcklund orbits of initial purely differential Lax operators. Our construction establishes the condition for commutativity of the additional-symmetry flows with the discrete Darboux–Bäcklund transformations of cKP hierarchies leading to a new derivation of the string-equation constraint in matrix models. © 1997 Published by Elsevier Science B.V.

1. Introduction

Relations between integrable models and conformal symmetries have been studied intensely since the first early signs of their interconnection showed up in the literature in seventies [1]. More recently, the KdV hierarchy formulation of nonperturbative 2-d quantum gravity [2] in the framework of (multi-)matrix models prompted more studies in this field. The subsequent work pointed out the nonisospectral symmetry origin of the pertinent Virasoro constraints on the string partition function but remained mostly limited to the KdV-like reduction of the KP hierarchy since it was dealing with the double scaling limit of the matrix models [3].

Quite recently a new class of integrable systems appeared both in mathematical literature [4] and independently in physics literature [5], where the motivation came from Toda field theory and discrete matrix models. These systems belong to the class of the so-called constrained KP hierarchies (cKP) as they are

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obtained by a symmetry reduction (which generalizes the KdV type of reduction) from the underlying general (unconstrained) KP hierarchy. The cKP hierarchies contain a large number of interesting hierarchies of soliton equations.

We address here the issue of formulating the additional nonisospectral Virasoro symmetry structure for the cKP hierarchies. This amounts to solving the problem of compatibility of the constraints with the additional nonisospectral symmetries of the original KP hierarchy. First, we show that the Virasoro algebra formulated according to the standard approach to KP additional symmetries [6] is broken by the cKP constraints down to its $sl(2)$ subalgebra (containing Galilean and scaling symmetries). Next, we show how to recover the full Virasoro symmetry (for the Virasoro generators $\mathcal{L}_n, n \geqslant -1$) by adding to the standard Virasoro generators "ghost" symmetry flows related to the plethora of (adjoint) eigenfunctions characteristic for the cKP Lax operator formulation.

We also discuss a special case of cKP hierarchies – the so-called truncated KP hierarchies obtained as Darboux–Bäcklund (DB) orbits of initial purely differential Lax operators. Application of our construction establishes the condition for commutativity of additional-symmetry flows with the discrete Darboux–Bäcklund transformations. This condition sheds new light on the derivation of the string-equation constraint (string condition) for matrix models. Details of calculations will appear elsewhere [7].

2. Background on KP hierarchy

We use the calculus of the pseudodifferential operators to describe the KP hierarchy of nonlinear evolution equations. In what follows the operator D is such that $[D, f] = f'$ with $f' = \partial f = \partial f/\partial x$ and it satisfies the generalized Leibniz rule (Eq. (A.1) from the Appendix).

The main object here is the pseudo-differential Lax operator Q

$$Q = D^r + \sum_{j=0}^{r-2} v_j D^j + \sum_{i\geqslant 1} u_i D^{-i} \tag{1}$$

of a generalized KP hierarchy (here "generalized" refers to the fact that Q is an rth order operator with $r \geqslant 1$, see also Ref. [8]). The associated Lax equations (with $x \equiv t_1$),

$$\frac{\partial}{\partial t_l} Q = [Q_+^{l/r}, Q], \qquad l = 1,2,\ldots, \tag{2}$$

describe isospectral deformations of Q. In (2) and below, the subscripts $(\pm)$ of pseudo-differential operators indicate purely differential/pseudo-differential parts. Commutativity of the isospectral flows $\partial/\partial t_l$ (2) is then assured by the Zakharov–Shabat equations. One can also represent the Lax operator in terms of the dressing operator $W = 1 + \sum_1^\infty w_n D^{-n}$ through $Q = WD^r W^{-1}$. In this framework Eq. (2) is equivalent to the so-called Wilson–Sato equation,

$$\frac{\partial}{\partial t_l} W = -(WD^l W^{-1})_- W. \tag{3}$$

For a given Lax operator Q, which satisfies Sato's flow equation (2), we call the function Φ (Ψ), whose flows are given by the expression[6],

$$\frac{\partial \Phi}{\partial t_l} = Q_+^{l/r}(\Phi), \qquad \frac{\partial \Psi}{\partial t_l} = -(Q^*)_+^{l/r}(\Psi), \qquad l = 1,2,\ldots, \tag{4}$$

[6] For any (pseudo-)differential operator A and a function f, the symbol $A(f)$ will indicate application (action) of A on f as opposed to the symbol Af meaning just operator product of A with the zero-order (multiplication) operator f.

an *(adjoint) eigenfunction* of Q. In (4) we have introduced an operation of conjugation, defined by simple rules $D^* = -D$ and $(AB)^* = B^*A^*$. An eigenfunction, which in addition also satisfies the spectral equations $Q\psi(\lambda, t) = \lambda\psi(\lambda, t)$ is called *Baker-Akhiezer (BA) function*.

3. Additional symmetries for the KP hierarchy

The KP hierarchy has an infinite set of commuting symmetries associated with the isospectral flows described above (Eq. (2)). However, the group of symmetries of the standard KP hierarchy is known to be much bigger. The extra symmetries are called "nonisospectral" or "additional" symmetries. A convenient approach to deal with symmetries of the integrable hierarchies of equations was developed by Orlov and Schulman (see Refs. [6,9,10]) and this is the approach we will use here. Other important contributions to the subject of additional symmetries for the KP hierarchy were made by Fuchssteiner [11] and Chen et al. [12]. See also Ref. [13] for the related discussion of the AKNS model, Ref. [14] for the truncated KP hierarchy and Ref. [15] for treatment of the generalized matrix hierarchies.

Let M be an operator "canonically conjugated" to Q such that

$$[Q, M] = \mathbb{1}, \qquad \frac{\partial}{\partial t_l} M = [Q_+^{l/r}, M]. \tag{5}$$

The M-operator can be expressed in terms of dressing of the "bare" $M^{(0)}$ operator

$$M^{(0)} = \sum_{l\geqslant 1} \frac{l}{r} t_l D^{l-r} = X_{(r)} + \sum_{l\geqslant 1} \frac{l+r}{r} t_{r+l} D^l, \qquad X_{(r)} \equiv \sum_{l=1}^{r} \frac{l}{r} t_l D^{l-r}, \tag{6}$$

conjugated to the "bare" Lax operator $Q^{(0)} = D^r$. The dressing gives

$$M = WM^{(0)}W^{-1} = WX_{(r)}W^{-1} + \sum_{l\geqslant 1} \frac{l+r}{r} t_{r+l} Q^{l/r} = \sum_{l\geqslant 0} \frac{l+r}{r} t_{r+l} Q_+^{l/r} + M_-, \tag{7}$$

$$M_- = WX_{(r)}W^{-1} - t_r - \sum_{l\geqslant 1} \frac{l+r}{r} t_{r+l} \frac{\partial W}{\partial t_l} \cdot W^{-1} \tag{8}$$

where in (8) we used Eqs. (3). Note that $X_{(r)}$ is a pseudo-differential operator satisfying $[D^r, X_{(r)}] = \mathbb{1}$.

The so-called *additional (nonisospectral) symmetries* [6,9] are defined as vector fields on the space of KP Lax operators (1) or, alternatively, on the dressing operator through their flows as follows,

$$\bar{\partial}_{k,n} Q = -[(M^n Q^k)_-, Q] = [(M^n Q^k)_+, Q] + nM^{n-1}Q^k, \qquad \bar{\partial}_{k,n} W = -(M^n Q^k)_- W. \tag{9}$$

The additional flows commute with the usual KP hierarchy flows given in (2). But they do not commute among themselves, instead they form the $\mathbf{W}_{1+\infty}$ algebra (see, e.g., Ref. [9]). One finds that the Lie algebra of operators $\bar{\partial}_{k,n}$ is isomorphic to the Lie algebra generated by $-z^n(\partial/\partial z)^k$. Especially for $n = 1$ this becomes an isomorphism to the Virasoro algebra $\bar{\partial}_{k,1} \sim -\mathcal{L}_{k-1}$, with $[\mathcal{L}_n, \mathcal{L}_k] = (n-k)\mathcal{L}_k$.

4. Constrained KP hierarchy and additional symmetry

We now turn to the main problem of this Letter, namely, compatibility of the additional Virasoro symmetry with the constraints defining the cKP hierarchy. We first introduce the symmetry constraints leading to the cKP hierarchy. Let ∂_{α_i} be vector fields, whose action on the standard KP Lax operator (with $r = 1$) $\mathcal{A} = D + \sum_{i=0}^{\infty} u_i D^{-i-1}$ is induced by the (adjoint) eigenfunctions Φ_i, Ψ_i of $\mathcal{A}$ through [4]

$$\partial_{\alpha_i}\mathcal{A} \equiv [\mathcal{A}, \Phi_i D^{-1}\Psi_i] \tag{10}$$

Let us recall the following fundamental property.

The vector fields ∂_{α_i} commute with the isospectral flows of the Lax operator $\mathcal{A}$,

$$[\partial_{\alpha_i}, \partial/\partial t_l]\mathcal{A} = 0, \qquad l = 1, 2, \ldots, \tag{11}$$

The constrained KP hierarchy (denoted as $\mathrm{cKP}_{r,m}$) is then obtained by identifying the "ghost" symmetry flow $\sum_{i=1}^{m} \partial_{\alpha_i}$ with the isospectral flow $\partial/\partial t_r$ of the original KP hierarchy.

Comparing (10) with Eq. (2) we find that for the Lax operator belonging to the $\mathrm{cKP}_{r,m}$ hierarchy we have $\mathcal{A}^r_- = \sum_{i=1}^{m} \Phi_i D^{-1}\Psi_i$. Hence we are led to the Lax operator $L = \mathcal{A}^r$ given by

$$L = L_+ + \sum_{i=1}^{m} \Phi_i D^{-1}\Psi_i = D^r + \sum_{l=0}^{r-2} v_l D^l + \sum_{i=1}^{m} \Phi_i D^{-1}\Psi_i \tag{12}$$

and subject to the Lax equation (2). Therefore, we parametrize the $\mathrm{cKP}_{r,m}$ hierarchy in terms of the Lax operator (12) and consider in what follows the operator M conjugated to L from (12). Note that the (adjoint) eigenfunctions Φ_i, Ψ_i of the original Lax operator $\mathcal{A}$ used in the above construction remain (adjoint) eigenfunctions for L (12) [16].

Applying the additional-symmetry flows (9) on L (12) for $n = 1$ we get

$$(\bar{\partial}_{k,1}L)_- = [(ML^k)_+, L]_- + (L^k)_-. \tag{13}$$

Using the simple identities (A.3) and (A.5) from the Appendix for the Lax operator (12), we are able to rewrite (13) as

$$(\bar{\partial}_{k,1}L)_- = \sum_{i=1}^{m}(ML^k)_+(\Phi_i)D^{-1}\Psi_i - \sum_{i=1}^{m}\Phi_i D^{-1}(ML^k)^*_+(\Psi_i) + \sum_{i=1}^{m}\sum_{j=0}^{k-1} L^{k-j-1}(\Phi_i)D^{-1}(L^*)^j(\Psi_i). \tag{14}$$

Here

$$L(\Phi_i) \equiv L_+(\Phi_i) + \sum_{j=1}^{m} \Phi_j \partial_x^{-1}(\Psi_j\Phi_i) \tag{15}$$

(and similarly for the adjoint counterpart) denotes action of L on Φ_i. Notice that $L^{k-j-1}(\Phi_i)$, $(L^*)^j(\Psi_i)$ are (adjoint) eigenfunctions of L (12). Hence, whereas the original L (12) belongs to the class of $\mathrm{cKP}_{r,m}$ hierarchies, the transformed Lax operator given by $\bar{\partial}_{k,1}L$ (cf. Eq. (14)) belongs to a *different* class – $\mathrm{cKP}_{r,m(k-1)}$ (for $k \geqslant 3$), since the number of eigenfunctions compared with formula (12) has increased.

For $k = 0, 1, 2$ the flow equations (14) can still be rewritten in the desired original $\mathrm{cKP}_{r,m}$ form,

$$(\partial_\tau L)_- = \sum_{i=1}^{m}(\partial_\tau\Phi_i)D^{-1}\Psi_i + \Phi_i D^{-1}(\partial_\tau\Psi_i), \tag{16}$$

with $\partial_\tau \equiv \bar{\partial}_{k,1}$ $(k = 0, 1, 2)$, where

$$\bar{\partial}_{0,1}\Phi_i = (M)_+(\Phi_i), \qquad \bar{\partial}_{0,1}\Psi_i = -(M)_+^*(\Psi_i), \tag{17}$$

$$\bar{\partial}_{1,1}\Phi_i = (ML)_+(\Phi_i) + \alpha\Phi_i, \qquad \bar{\partial}_{1,1}\Psi_i = -(ML)_+^*(\Psi_i) + \beta\Psi_i, \qquad \alpha + \beta = 1, \tag{18}$$

$$\bar{\partial}_{2,1}\Phi_i = (ML^2)_+(\Phi_i) + L(\Phi_i), \qquad \bar{\partial}_{2,1}\Psi_i = -(ML^2)_+^*(\Psi_i) + L^*(\Psi_i). \tag{19}$$

Note an ambiguity on the right hand sides of (18).

Since the additional flows satisfy an algebra $[\bar{\partial}_{l,1}, \bar{\partial}_{k,1}] = -(l-k)\bar{\partial}_{l+k-1,1}$ we have an isomorphism $\bar{\partial}_{k,1} \sim -\mathcal{L}_{k-1}$ with the Virasoro operators and Eqs. (17)-(19) contain the $sl(2)$ subalgebra generators $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1$.

However, for $\partial_\tau \equiv \bar{\partial}_{k,1}$ $k \geqslant 3$, Eq. (16) does not hold anymore due to absence of consistent definitions for $\bar{\partial}_{k,1}\Phi_i, \bar{\partial}_{k,1}\Psi_i$ generalizing (17)-(19) for higher k. Thus, it appears that the symmetry constraints behind the cKP hierarchies have broken the standard KP additional Virasoro symmetry down to its $sl(2)$ subalgebra.

To recover the complete Virasoro symmetry, our strategy will be to redefine the additional-symmetry generators. We first describe our technique for $k = 3$ in which case Eq. (13) contains a term

$$(L^3)_- = \sum_{i=1}^{m} \Phi_i D^{-1}(L^*)^2(\Psi_i) + \sum_{i=1}^{m} L(\Phi_i) D^{-1} L^*(\Psi_i) + \sum_{i=1}^{m} L^2(\Phi_i) D^{-1}\Psi_i \tag{20}$$

Note that the middle term in (20) is not consistent with the form of Eq. (16). At this point we recall that for the pseudo-differential operator

$$X \equiv \sum_{k=1}^{l} M_k D^{-1} N_k \tag{21}$$

with definitions (12) and (21) we find using identity (A.4) from the Appendix,

$$[X, L]_- = \sum_{k=1}^{l} [-L(M_k)D^{-1}N_k + M_k D^{-1}L^*(N_k)] + \sum_{i=1}^{m} [X(\Phi_i)D^{-1}\Psi_i - \Phi_i D^{-1}X^*(\Psi_i)]. \tag{22}$$

We now introduce the family of pseudo-differential operators of the same type as X in (21),

$$X_k^{(0)} \equiv \sum_{i=1}^{m}\sum_{j=0}^{k-1} L^{k-1-j}(\Phi_i)D^{-1}(L^*)^j(\Psi_i), \qquad k \geqslant 1, \tag{23}$$

$$X_k^{(1)} \equiv \sum_{i=1}^{m}\sum_{j=0}^{k-1} [j - \tfrac{1}{2}(k-1)] L^{k-1-j}(\Phi_i)D^{-1}(L^*)^j(\Psi_i), \qquad k \geqslant 1, \tag{24}$$

$$X_k^{(2)} \equiv \sum_{i=1}^{m}\sum_{j=0}^{k-1} [j^2 - j(k-1) + \tfrac{1}{6}(k-2)(k-1)] L^{k-1-j}(\Phi_i)D^{-1}(L^*)^j(\Psi_i), \qquad k \geqslant 1. \tag{25}$$

According to Eq. (11) the flows generated by (22) will commute with the isospectral flows (2) provided M_i, N_i are (adjoint) eigenfunctions, which will be the case in what follows.

Since $X_k^{(0)} = (L^k)_-$ for the cKP hierarchy [17] the operators from (23) generate the standard isospectral flows. We now investigate the role of remaining operators from Eqs. (24), (25) for the construction of nonisospectral flows within the cKP hierarchy. Considering first as an example operator $X_{k=2}^{(1)}$ from Eq. (24) we find that

$$[X_2^{(1)}, L]_- = -(L^3)_- + \tfrac{3}{2}\sum_{i=1}^{m}[\Phi_i D^{-1}(L^*)^2(\Psi_i) + L^2(\Phi_i)D^{-1}\Psi_i]$$
$$+\sum_{i=1}^{m}[X_2^{(1)}(\Phi_i)D^{-1}\Psi_i - \Phi_i D^{-1}X_2^{(1)}(\Psi_i)]. \tag{26}$$

Hence $[-(ML^3)_- + X_2^{(1)}, L]_-$ still has the form of (16). Therefore, it may be possible to find additional symmetries for the cKP models by combining the original $\bar{\partial}_{k,1}$ flows and the ghost symmetry flows (10) associated with operators of the type as in (24). This will work provided that the above construction yields the Virasoro generator $\mathcal{L}_2$ obeying the correct algebra with the unbroken $sl(2)$ generators found above in (17)–(19).

We now generalize the above manipulations to an arbitrary k using definitions (23)–(25). Acting on (23)–(25) with $\bar{\partial}_{\ell,1}$ for $\ell = 0, 1, 2$ and using

$$\bar{\partial}_{\ell,1}L^k(\Phi_i) = (ML^\ell)_+(L^k(\Phi_i)) + (k + \tfrac{1}{2}\ell)L^{k+\ell-1}(\Phi_i),$$
$$\bar{\partial}_{\ell,1}(L^*)^k(\Psi_i) = -(ML^\ell)_+^*((L^*)^k(\Psi_i)) + (k + \tfrac{1}{2}\ell)(L^*)^{k+\ell-1}(\Psi_i), \tag{27}$$

valid for $\ell = 0, 1, 2$ and $k \geqslant 0$, we get

$$\bar{\partial}_{\ell,1}X_k^{(0)} = [(ML^\ell)_+, X_k^{(0)}]_- + kX_{k+\ell-1}^{(0)}, \tag{28}$$
$$\bar{\partial}_{\ell,1}X_k^{(1)} = [(ML^\ell)_+, X_k^{(1)}]_- + (k - \ell + 1)X_{k+\ell-1}^{(1)}, \tag{29}$$
$$\bar{\partial}_{\ell,1}X_k^{(2)} = [(ML^\ell)_+, X_k^{(2)}]_- + [k - 2(\ell-1)]X_{k+\ell-1}^{(2)} - \tfrac{1}{6}[(\ell-1)^3 - (\ell-1)]X_{k+\ell-1}^{(0)}. \tag{30}$$

Here we recognize the structure of the $\mathbf{W}_{1+\infty}$ algebra under substitution $\ell \to \ell - 1$ (see, e.g., Ref. [18]). Let us now restrict our attention to the part of the algebra involving the $X_{k-1}^{(1)}$ operator from (24). We note that (22) and identity (A.5) from the Appendix enable us to obtain

$$[X_{k-1}^{(1)}, L]_- = \frac{k}{2}\sum_{i=1}^{m}[\Phi_i D^{-1}(L^*)^{k-1}(\Psi_i) + L^{k-1}(\Phi_i)D^{-1}\Psi_i] - (L^k)_-$$
$$+\sum_{i=1}^{m}[-\Phi_i D^{-1}(X_{k-1}^{(1)})^*(\Psi_i) + X_{k-1}^{(1)}(\Phi_i)D^{-1}\Psi_i]. \tag{31}$$

Putting together (14) and (31) yields our main result:

The correct additional-symmetry flows for the cKP hierarchies (12), spanning the Virasoro algebra, are given by

$$\partial_k^* L \equiv [-(ML^k)_- + X_{k-1}^{(1)}, L], \tag{32}$$

i.e., with the isomorphism $\mathcal{L}_{k-1} \sim -(ML^k)_- + X_{k-1}^{(1)}$, where $X_{k-1}^{(1)}$ are defined in (24). Accordingly, on dressing operators and BA functions the flows (32) read

$$\partial_k^* W = [-(ML^k)_- + X_{k-1}^{(1)}]W, \qquad \partial_k^*\psi(t,\lambda) = [-(ML^k)_- + X_{k-1}^{(1)}](\psi(t,\lambda)). \tag{33}$$

First, observe that the flows (32) preserve the $\mathrm{cKP}_{r,m}$ form (12). Indeed, $(\partial_k^* L)_-$ can be cast in the form of (16) with

$$\partial_k^* \Phi_i = (ML^k)_+(\Phi_i) + \frac{k}{2} L^{k-1}(\Phi_i) + X_{k-1}^{(1)}(\Phi_i),$$

$$\partial_k^* \Psi_i = -(ML^k)_+^*(\Psi_i) + \frac{k}{2}(L^*)^{k-1}(\Psi_i) - (X_{k-1}^{(1)})^*(\Psi_i). \tag{34}$$

Taking into account that $X_{i-1}^{(1)} = 0$ for $i = 0,1,2$ we see that Eqs. (34) reproduce (17)-(19) (with ambiguity on the right hand side of (18) removed by fixing $\alpha = \beta = 1/2$). Hence $\partial_\ell^* = \bar{\partial}_{\ell,1}$ for $\ell = 0,1,2$.

Secondly, we note that the modified additional symmetry flows defined by (32) commute with the isospectral flows (2) according to (10), (11).

The remaining question is whether they form a closed algebra. Indeed, using identity (29) we arrive at the fundamental commutation relations for $\ell = 0,1,2$ and any $k \geqslant 0$,

$$[\partial_\ell^*, \partial_k^*] L = (k - \ell)\partial_{k+\ell-1}^* L. \tag{35}$$

This discussion shows that $[\mathcal{L}_i, \mathcal{L}_k] = (i-k)\mathcal{L}_{i+k}$ for $i = -1,0,1$ ($sl(2)$ generators) and arbitrary k, where $\mathcal{L}_{k-1} \sim -\partial_k^*$. Since according to (32) the generator $\mathcal{L}_2$ is associated with $X_2^{(1)} - (ML^3)_-$, all higher Virasoro operators can be obtained recursively from

$$\mathcal{L}_{n+1} = \frac{-1}{(n-1)}[\mathcal{L}_n, \mathcal{L}_1], \qquad n \geqslant 2. \tag{36}$$

Then Eq. (35) implies that $\mathcal{L}_n$ with $n \geqslant 3$ may differ from the generators given by the flows $\partial_{n+1}^* \sim -(ML^{n+1})_- + X_n^{(1)}$ defined in (32) at most by flows commuting with the $sl(2)$ additional symmetry generators, i.e., by ordinary isospectral flows. Therefore, we can now easily show by induction that $\mathcal{L}_k, k \geqslant -1$, obtained in the above way form a closed Virasoro algebra up to irrelevant terms containing ordinary isospectral flows.

5. Darboux-Bäcklund transformations of cKP hierarchies. Truncated KP hierarchies

Let Φ be an eigenfunction of L defining a Darboux-Bäcklund transformation, i.e.,

$$\frac{\partial}{\partial t_l}\Phi = L_+^{l/r}(\Phi), \qquad \widetilde{L} = (\Phi D \Phi^{-1}) L (\Phi D^{-1} \Phi^{-1}), \qquad \widetilde{W} = (\Phi D \Phi^{-1}) W D^{-1} \tag{37}$$

Then the DB-transformed M operator (cf. (7)) acquires the form

$$\widetilde{M} = (\Phi D \Phi^{-1}) M (\Phi D^{-1} \Phi^{-1}) = \sum_{l \geqslant 0} \frac{l+r}{r} t_{r+l} \widetilde{L}_+^{l/r} + \widetilde{M}_-, \tag{38}$$

$$\widetilde{M}_- = \widetilde{W}\widetilde{X}_{(r)}\widetilde{W}^{-1} - t_r - \sum_{l \geqslant 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l}\widetilde{W} \cdot \widetilde{W}^{-1}, \tag{39}$$

where $\widetilde{X}_{(r)} = D X_{(r)} D^{-1}$ with $X_{(r)}$ as in (6). Clearly $\widetilde{X}_{(r)}$, like $X_{(r)}$, is also admissible as canonically conjugated to D^r.

In particular, for L belonging to a cKP hierarchy (12) we consider a special class of DB transformations (37) which preserve the constrained cKP form of L,

$$\widetilde{L} = T_a L T_a^{-1} = \widetilde{L}_+ + \sum_{i=1}^m \widetilde{\Phi}_i D^{-1} \widetilde{\Psi}_i, \qquad T_a \equiv \Phi_a D \Phi_a^{-1}, \tag{40}$$

$$\widetilde{\Phi}_a = T_a L(\Phi_a), \qquad \widetilde{\Psi}_a = \Phi_a^{-1}, \tag{41}$$

$$\tilde{\Phi}_i = T_a(\Phi_i), \qquad \tilde{\Psi}_i = T_a^{-1*}\Psi_i = -\Phi_a^{-1}\partial_x^{-1}(\Psi_i\Phi_a), \qquad i \neq a, \tag{42}$$

where the DB-generating $\Phi \equiv \Phi_a$ coincides with one of the eigenfunctions appearing in formula (12) of the initial L.

Let us consider the generic class of DB orbits on $\mathrm{cKP}_{r,m}$ consisting of k_1 transformations generated by Φ_1, followed by k_2 transformations in Φ_2 direction and so on, until k_m transformations in Φ_m direction. Repeated use of a composition formula for Wronskians (see Eqs. (A.10)-(A.12) from the Appendix) leads us to the following explicit expressions for the successive eigenfunctions and the τ-function obtained after $\sum_{a=1}^{m} k_a$ steps of successive DB transformations (see Ref. [19] and also Ref. [20]),

$$\begin{aligned}\Phi_a^{(k_1,\ldots,k_a,0,\ldots)} &= \prod_{b=a}^{1}\prod_{j=0}^{k_b-1} T_b^{(k_1-1,\ldots,k_b-1-j,0,\ldots)}\left((L^{(0)})^{k_a}(\Phi_a^{(0)})\right)\\ &= \frac{W[\chi_1^{(0)},\ldots,\chi_1^{(k_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(k_a-1)},\chi_a^{(k_a)}]}{W[\chi_1^{(0)},\ldots,\chi_1^{(k_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(k_a-1)}]},\end{aligned} \tag{43}$$

$$\frac{\tau^{(k_1,\ldots,k_m)}}{\tau^{(0)}} = \prod_{a=m}^{1}\prod_{j=0}^{k_a-1}\Phi_a^{(k_1,\ldots,k_{a-1},k_a-1-j,0,\ldots)} = W[\chi_1^{(0)},\ldots,\chi_1^{(k_1-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(k_m-1)}], \tag{44}$$

where the upper indices in parentheses indicate the order of the corresponding DB step, the zero index referring to the "initial" cKP Lax operator, and where we have employed the short-hand notations:

$$T_a^{(i_1,\ldots,i_m)} \equiv \Phi_a^{(i_1,\ldots,i_m)}D(\Phi_a^{(i_1,\ldots,i_m)})^{-1}, \qquad \chi_a^{(s)} \equiv (L^{(0)})^s(\Phi_a^{(0)}), \qquad a = 1,\ldots,m. \tag{45}$$

As seen from (40)-(42) and (43), the DB orbit $L^{(k)} = (L^{(k)})_+ + \sum_{i=1}^{m}\Phi_i^{(k)}D^{-1}\Psi_i^{(k)}$ of $\mathrm{cKP}_{r,m}$, starting from a purely differential initial $L^{(0)} = (L^{(0)})_+$, defines a class of *truncated* $\mathrm{cKP}_{r,m}$ hierarchies where the m adjoint eigenfunctions $\Psi_i \equiv \Psi_i^{(k)}$ are not independent of the m eigenfunctions $\Phi_i \equiv \Phi_i^{(k)}$ since both are parametrized in terms of m initial eigenfunctions $\Phi_i^{(0)}$ only.

As a simple example of truncated cKP hierarchies, consider formulas (43), (44) for the DB orbit of the $\mathrm{cKP}_{1,m}$ hierarchy (the so-called "multi-boson" reduction of the general KP hierarchy) starting from a "free" initial $L^{(0)} = D$. In this case we have to substitute in (43), (44),

$$\chi_i^{(s)} \equiv \partial^s\Phi_i^{(0)}, \qquad \Phi_i^{(0)} = \int_\Gamma \mathrm{d}\lambda\,\phi_i^{(0)}(\lambda)\exp\Bigl(\sum_{r\geqslant 1}\lambda^r t_r\Bigr), \tag{46}$$

with arbitrary "densities" $\phi_i^{(0)}(\lambda)$ (and with appropriate contour Γ such that the λ-integrals exist). A special feature of truncated $\mathrm{cKP}_{1,m}$ is that their dressing operators are truncated (having only a finite number of terms in the pseudo-differential expansion, cf. Ref. [14]),

$$W^{(k_1,\ldots,k_m)} = \prod_{a=m}^{1}\prod_{j=0}^{k_a-1}T_a^{(k_1-1,\ldots,k_a-1-j,0,\ldots)}D^{-N_m} = \sum_{j=0}^{N_m} w_j^{(k_1,\ldots,k_m)}D^{-j}, \qquad N_m \equiv \sum_{a=1}^{m}k_a, \tag{47}$$

where notations (45) were used.

The particular case $m = 1$ of (40)-(44) yields

$$L^{(k+1)} = (\Phi^{(k)}D\Phi^{(k)^{-1}})L^{(k)}(\Phi^{(k)}D^{-1}\Phi^{(k)^{-1}}) = D + \Phi^{(k+1)}D^{-1}\Psi^{(k+1)}, \tag{48}$$

$$\Phi^{(k+1)} = \Phi^{(k)}(\ln\Phi^{(k)})'' + (\Phi^{(k)})^2\Psi^{(k)}, \qquad \Psi^{(k+1)} = (\Phi^{(k)})^{-1}, \tag{49}$$

$$\Phi^{(n)} = \frac{W_{n+1}[\phi,\partial\phi,\ldots,\partial^n\phi]}{W_n[\phi,\partial\phi,\ldots,\partial^{n-1}\phi]}, \qquad \tau^{(n)} = W_n[\phi,\partial\phi,\ldots,\partial^{n-1}\phi], \tag{50}$$

where

$$\phi \equiv \Phi^{(0)} = \int \mathrm{d}\lambda \phi^{(0)}(\lambda) \exp\Big(\sum_{r=1}^{\infty} t_r \lambda^r\Big). \tag{51}$$

The hierarchies given by (48) are generalizations of the Burgers–Hopf hierarchy defined by $L^{(1)} = D + \phi(\ln\phi)'' D^{-1}\phi^{-1}$.

6. Additional symmetries versus DB transformations for cKP hierarchies. String condition

With the help of identities (A.6)–(A.9) from the Appendix we find the following explicit form of the DB transformation of the operators $X^{(1)}_{k-1}$ (24),

$$T_a X^{(1)}_{k-1} T_a^{-1} = \widetilde{X}^{(1)}_{k-1} - (\widetilde{L}^{(a)})^{k-1}_{-} + [T_a(X^{(1)}_{k-1} + \tfrac{1}{2}kL^{k-1})(\Phi_a)] D^{-1}\Phi_a^{-1}, \tag{52}$$

$$(\widetilde{L}^{(a)})^{k-1}_{-} \equiv \sum_{j=0}^{k-2} \widetilde{L}^{k-j-2}(\widetilde{\Phi}_a) D^{-1} (\widetilde{L}^{*})^{j} (\widetilde{\Psi}_a). \tag{53}$$

Here $\widetilde{L}, T_a$ are as in (40) and the DB-transformed $\widetilde{X}^{(1)}_{k-1}$ have the same form as $X^{(1)}_{k-1}$ in (24) with all (adjoint) eigenfunctions substituted with their DB-transformed counterparts as in (40)–(42). Also notice that in the particular case of $\mathrm{cKP}_{r,1}$ hierarchies $(\widetilde{L}^{(a)})^{k-1}_{-}$ (53) coincides with the (pseudo-differential part of the power of the) full $\mathrm{cKP}_{r,1}$ Lax operator (cf. Eq. (12) for $m = 1$ and (A.5)).

Taking into account (40)–(42) and (52), (53) we conclude that:

The additional-symmetry flows (32) for $\mathrm{cKP}_{r,1}$ hierarchies (Eq. (12) with $m = 1$) commute with the Darboux–Bäcklund transformations (40) preserving the form of $\mathrm{cKP}_{r,1}$, up to shifting of (32) by ordinary isospectral flows. Explicitly we have

$$\partial_k^{*} \widetilde{L} = -[(\widetilde{M}\widetilde{L}^{k})_{-} - \widetilde{X}^{(1)}_{k-1}, \widetilde{L}] + \frac{\partial \widetilde{L}}{\partial t_{k-1}}. \tag{54}$$

Eq. (54) shows that the additional-symmetry flows (32) are well-defined for all $\mathrm{cKP}_{r,1}$ Lax operators belonging to a given DB orbit of successive DB transformations. Notice that it is precisely the class of (truncated) $\mathrm{cKP}_{r,1}$ hierarchies which is relevant for the description of discrete (multi-)matrix models [21,16,19].

Motivated by applications to (multi-)matrix models (see Ref. [7]), one can require invariance of cKP hierarchies under some of the additional-symmetry flows, e.g., under the lowest one $\partial_0^{*} \equiv \bar{\partial}_{0,1}$ known as "string-equation" constraint (string condition) in the context of the (multi-)matrix models,

$$\partial_0^{*} L = 0 \;\rightarrow\; [M_{+}, L] = -\mathbb{1}, \qquad \partial_0^{*}\Phi = 0 \;\rightarrow\; M_{+}\Phi = 0. \tag{55}$$

Eqs. (55), using the second Eq. (5), (7) and the first Eq. (34) for $k = 0$, lead to the following constraints on L (12) and its DB-generating eigenfunction Φ, respectively,

$$\sum_{l\geqslant 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} L + [t_1, L]\delta_{r,1} = -\mathbb{1}, \tag{56}$$

$$\Big(\sum_{l\geqslant 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} + t_r\Big)\Phi = 0. \tag{57}$$

Recall now the formula (44) for the τ-function of the $\mathrm{cKP}_{r,m}$ hierarchy (12). Noticing that the eigenfunctions $\Phi^{(k)}$ of the DB-transformed Lax operators $L^{(k)}$ satisfy the *same* constraint Eq. (57) irrespective of the DB-step k, we arrive at the following result ("string-equation" constraint on the τ-functions):

The Wronskian τ-functions (44) of $\mathrm{cKP}_{r,m}$ hierarchies (12), invariant under the lowest additional symmetry flow (55), satisfy the constraint equation

$$\left(\sum_{l\geqslant 1}\frac{l+r}{r}t_{r+l}\frac{\partial}{\partial t_l}+nt_r\right)\frac{\tau^{(n)}}{\tau^{(0)}}=0. \tag{58}$$

As the simplest illustration, consider the discrete one-matrix model corresponding to the generalized Burgers–Hopf hierarchy, i.e., to the chain of the Lax operators connected via DB transformations as described in Eqs. (48), (49), but with the *additional restriction* on $\phi\equiv\Phi^{(0)}$ (51) (coming from the orthogonal polynomial formalism),

$$\phi=\int \mathrm{d}\lambda\exp\left(\sum_{r=1}^{\infty}t_r\lambda^r\right),\qquad \text{i.e., } \phi^{(0)}(\lambda)=1. \tag{59}$$

The initial "free" eigenfunction (59) obeys the constraint Eq. (57) (for $r=1$) and, therefore, Eq. (58) (with $r=1$) for the $\tau^{(n)}$ as in (50) yields precisely the "string-equation" in the one-matrix model,

$$\mathcal{L}_{-1}^{(n)}W_n[\phi,\partial\phi,\dots,\partial^{n-1}\phi]=0,\qquad \mathcal{L}_{-1}^{(n)}\equiv\sum_{k=2}^{\infty}kt_k\frac{\partial}{\partial t_{k-1}}+nt_1 \tag{60}$$

Furthermore, as one can check directly [7], the Wronskian τ-function (second Eq. (50)) with ϕ restricted as in (59) automatically satisfies all higher Virasoro constraints. Thus, we conclude that for the particular class of cKP hierarchies – the generalized Burgers–Hopf hierarchies (48)–(51), invariance under the lowest additional-symmetry flow automatically triggers invariance under all higher additional-symmetry flows as well.

Appendix A. Technical identities

We list here for convenience a number of useful technical identities, which have been used extensively throughout the text.

We work with calculus of pseudo-differential operators based on the generalized Leibniz rule,

$$D^n f=\sum_{j=0}^{\infty}\binom{n}{j}(\partial^j f)D^{n-j}. \tag{A.1}$$

For an arbitrary pseudo-differential operator A we have the following identity,

$$(\chi D\chi^{-1}A\chi D^{-1}\chi^{-1})_+=\chi D\chi^{-1}A_+\chi D^{-1}\chi^{-1}-\chi\partial_x(\chi^{-1}A_+(\chi))D^{-1}\chi^{-1}, \tag{A.2}$$

where A_+ is the differential part of $A=A_++A_-=\sum_{i=0}^{\infty}A_iD^i+\sum_{-\infty}^{-1}A_iD^i$. For a purely differential operator K and arbitrary functions f,g we have an identity

$$[K,fD^{-1}g]_-=K(f)D^{-1}g-fD^{-1}K^*(g). \tag{A.3}$$

Another useful technical identity involves a product of two pseudo-differential operators of the form $X_i = f_i D^{-1} g_i$, $i = 1,2$,

$$X_1 X_2 = X_1(f_2) D^{-1} g_2 + f_1 D^{-1} X_2^*(g_1), \tag{A.4}$$

where $X_1(f_2) = f_1 \partial_x^{-1}(g_1 f_2)$, etc. From the above identity follows the relation [17]

$$(L^k)_- = \sum_{i=1}^{m} \sum_{j=0}^{k-1} L^{k-j-1}(\Phi_i) D^{-1} (L^*)^j (\Psi_i) \tag{A.5}$$

for the cKP Lax operator (12).

Let us also list some useful identities involving Darboux–Bäcklund-like transformation of pseudo-differential operators of the X_i-form above,

$$T_a(\Phi_a D^{-1} N) T_a^{-1} = (\Phi_a^2 N) D^{-1} \Phi_a^{-1}, \tag{A.6}$$

$$T_a(M D^{-1} \Psi_a) T_a^{-1} = \widetilde{M} D^{-1} (\widetilde{L}^*(\widetilde{\Psi}_a)) + \{T_a(M \partial_x^{-1}(\Psi_a \Phi_a))\} D^{-1} \Phi_a^{-1}, \tag{A.7}$$

$$T_a(M D^{-1} N) T_a^{-1} = \widetilde{M} D^{-1} \widetilde{N} + \{T_a(M \partial_x^{-1}(N \Phi_a))\} D^{-1} \Phi_a^{-1}, \tag{A.8}$$

$$(\widetilde{L}^*)^s (\widetilde{\Psi}_a) = -\Phi_a^{-1} \partial_x^{-1} (\Phi_a (L^*)^{s-1} (\Psi_a)), \tag{A.9}$$

where Φ_a is one of the eigenfunctions of a cKP Lax operator L (12) and

$$T_a \equiv \Phi_a D \Phi_a^{-1}, \qquad \widetilde{\Psi}_a = \Phi_a^{-1},$$

$$\widetilde{M} \equiv T_a(M) = \Phi_a \partial_x (\Phi_a^{-1} M), \qquad \widetilde{N} \equiv T_a^{-1*}(N) = -\Phi_a^{-1} \partial_x^{-1} (\Phi_a N).$$

Finally, let us recall the following important composition formula for Wronskians [22],

$$T_k T_{k-1} \dots T_1(f) = \frac{W_k(f)}{W_k}, \tag{A.10}$$

where

$$T_j = \frac{W_j}{W_{j-1}} D \frac{W_{j-1}}{W_j} = \left[D + \left(\ln \frac{W_{j-1}}{W_j} \right)' \right], \qquad W_0 = 1, \tag{A.11}$$

$$W_k \equiv W_k[\psi_1, \dots, \psi_k] = \det \| \partial_x^{i-1} \psi_j \|, \qquad W_{k-1}(f) \equiv W_k[\psi_1, \dots, \psi_{k-1}, f]. \tag{A.12}$$

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